

Generalised cosine functions, basis and regularity properties

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3rd November 2015

Abstract

We examine regularity and basis properties of the family of rescaled p -cosine functions. We find sharp estimates for their Fourier coefficients. We then determine two thresholds, $p_0 < 2$ and $p_1 > 2$, such that this family is a Schauder basis of $L_s(0, 1)$ for all $s > 1$ and $p \in [p_0, p_1]$.

1 Introduction

The contents of this paper can be summarised as follows. Consider a continuous 2-periodic function $f : \mathbb{R} \rightarrow \mathbb{C}$. Denote by \mathcal{F} the family of rescalings $\mathcal{F} = \{f(nx)\}_{n \in \mathbb{N}}$. When does \mathcal{F} form a Schauder basis of $L_s \equiv L_s(0, 1)$ for all $s > 1$? This question can be traced back to a 1945 note by Arne Beurling [1]. However, quite remarkably, there are still a number of open problems associated to it. As it turns, finding a concrete answer can be extremely difficult, even for apparently simple functions f .

In a series of recent papers the above question has been addressed for the particular choice $f(x) = \sin_p(\pi_p x)$, the p -sine functions. Let $p > 1$. Let the increasing function $F_p : [0, 1] \rightarrow [0, \frac{\pi_p}{2}]$ be defined by means of the integral

$$(1) \quad F_p(y) := \int_0^y (1 - t^p)^{-\frac{1}{p}} dt$$

where

$$\pi_p := 2F_p(1) = \frac{2\pi}{p \sin(\frac{\pi}{p})}.$$

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Denote the inverse of F_p by \sin_p , which is increasing in the segment $[0, \frac{\pi_p}{2}]$. Extend to the whole of \mathbb{R} by means of the rules

$$(2) \quad \sin_p(-x) = -\sin_p(x) \quad \text{and} \quad \sin_p\left(\frac{\pi_p}{2} - x\right) = \sin_p\left(\frac{\pi_p}{2} + x\right),$$

which makes it $2\pi_p$ -periodic and continuous in \mathbb{R} . The choice $p = 2$ corresponds to the standard trigonometric setting $\sin_2 \equiv \sin$, $\pi_2 = \pi$ and in this case \mathcal{F} is a Schauder basis of L_s for all $s > 1$ as a consequence of Fourier's Theorem.

The study of the p -sine functions originated in the context of the one-dimensional p -Laplacian non-linear eigenvalue problem and dates back to the work of Elbert [10] and Ôtani [14]. Their basis properties were first examined in [2], where it was announced that the family $\{\sin_p(n\pi_p \cdot)\}_{n \in \mathbb{N}}$ forms a Schauder basis of L_s for all $s > 1$ and $p \geq \frac{12}{11}$. Further development in this respect were settled in [5], [6] and [4]. Currently we know that this family is a Schauder basis of L_s for all $s > 1$ and $p > \tilde{p}_0$, and also a Riesz basis of L_2 for $p \in (\hat{p}_0, \tilde{p}_0]$, where $\tilde{p}_0 \approx 1.087$ and $\hat{p}_0 \approx 1.044$ satisfy complicated identities involving hypergeometric functions.

Let

$$(3) \quad \cos_p x := \frac{d}{dx} \sin_p x \quad \forall x \in \mathbb{R}$$

and set $f(x) = \cos_p(\pi_p x)$, the p -cosine functions. From the various results established in the recent paper [7], it follows that $\mathcal{F} \cup \{1\}$ is a Schauder basis of L_s for all $s > 1$ and $p \in (p_0^\dagger, 2]$ where $p_0^\dagger \approx 1.75$. In the present work we establish that this basis property in fact holds true for p in a wider segment. To be precise, we show the following.

Theorem 1. *There exist $p_0 < \frac{3}{2}$ and $p_1 > \frac{11}{5}$, such that $\{\cos_p(n\pi_p \cdot)\}_{n=0}^\infty$ is a Schauder basis of L_s for all $s > 1$ and $p \in [p_0, p_1]$.*

The constants p_0 and p_1 will be given analytically as the zeros of corresponding equations involving the parameter p . Their approximated values turn out to be $p_0 \approx 1.46$ and $p_1 \approx 2.43$.

The proof of Theorem 1 is naturally divided into the cases $1 < p < 2$ and $p > 2$. The different parts of the paper follow this division. In Section 2 we collect various properties of the p -trigonometric functions which will be useful later on. In Section 3 we establish precise upper bounds on the asymptotic behaviour of the Fourier coefficients of $\cos_p(\pi_p \cdot)$. In Section 4 we recall the framework for determining invertibility of the change of coordinates map between the families $\{\cos(n\pi \cdot)\}_{n=0}^\infty$ and $\{\cos_p(n\pi_p \cdot)\}_{n=0}^\infty$. In Section 5 we assemble the proof of Theorem 1, by combining the crucial criterion (12) of Section 4 with the estimates of Section 3. In the final Section 6 we describe the relation between the results announced here and other existing work.

2 The generalised trigonometric functions

We begin by recalling various elementary properties of the p -cosine functions. A more complete account on this matter can be found in [5, Section 2] and [9, Chapter 2].

Throughout we shall assume that $1 < p < \infty$. Note that π_p is a decreasing function, smooth in $p > 1$, such that

$$\begin{cases} \pi_p \rightarrow \infty & p \rightarrow 1^+ \\ \pi_p = \pi & p = 2 \\ \pi_p \rightarrow 2 & p \rightarrow \infty. \end{cases}$$

Here and everywhere below we write $p' := p/(p-1)$. According to [5, (2.3)], we know that

$$(4) \quad p' \pi_{p'} = p \pi_p.$$

From (2) and (3) it immediately follows that \cos_p is $2\pi_p$ -periodic,

$$\cos_p(x) = \cos_p(-x) \quad \text{and} \quad \cos_p\left(x + \frac{\pi_p}{2}\right) = -\cos_p\left(x - \frac{\pi_p}{2}\right) \quad \forall x \in \mathbb{R}.$$

Moreover, setting $y = \sin_p(x)$ for $x \in [0, \pi_p/2]$ in the formula for the derivative of the inverse function of (1), gives

$$(5) \quad \cos_p(x) = (1 - y^p)^{1/p} = (1 - \sin_p(x)^p)^{1/p}.$$

Thus, \cos_p is decreasing in $(0, \pi_p/2]$, $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. In fact we have,

$$|\sin_p x|^p + |\cos_p x|^p = 1 \quad \forall x \in \mathbb{R}.$$

See [5, (2.7)].

Lemma 1. *For all $x \in [0, \frac{1}{2})$,*

a.

$$\cos_p(\pi_p x) = \sin_{p'}\left(\pi_{p'}\left(\frac{1}{2} - x\right)\right)^{p'-1}$$

b.

$$\frac{d}{dx} \cos_p(x) = -\sin_p(x)^{p-1} \cos_p(x)^{2-p}$$

c.

$$\frac{d^2}{dx^2} \cos_p(x) = \sin_p(x)^{p-2} \cos_p(x)^{3-2p} [2 - p - \cos_p(x)^p].$$

Proof. The calculations leading to “a” and “b” can be found in the proofs of [5, Proposition 2.2] and [5, Proposition 2.1], respectively. From (5) we get

$$\begin{aligned}\frac{d^2}{dx^2} \cos_p(x) &= (2-p) \sin_p(x)^{2p-2} \cos_p(x)^{3-2p} - (p-1) \sin_p(x)^{p-2} \cos_p(x)^{3-p} \\ &= \sin_p(x)^{p-2} \cos_p(x)^{3-2p} [(2-p) \sin_p(x)^p - (p-1) \cos_p(x)^p],\end{aligned}$$

which is “c”. \square

The following inequalities will be important below.

Lemma 2. *Let $1 < p \leq q < \infty$ and $x \in [0, \frac{1}{2}]$. Then*

$$a. \sin_p(\pi_p x) \geq \sin_q(\pi_q x)$$

$$b. \cos_p(\pi_p x) \leq \cos_q(\pi_q x).$$

Proof. Statement “a” is [5, Corollary 4.4-(iii)].

Let us show “b”. A direct evaluation at $x = 0$ and $x = 1/2$ gives equality for all p and q at these points, so these two cases are immediate. Let $x \in (0, \frac{1}{2})$ be fixed. Since p' is decreasing in $p > 1$, from part “a” it follows that

$$\frac{d}{dp} \sin_{p'} \left(\pi_{p'} \left(\frac{1}{2} - x \right) \right) \geq 0 \quad \forall p \in (1, \infty).$$

Note that, $0 < \sin_{p'}(\pi_{p'}(\frac{1}{2} - x)) < 1$ and hence $\ln(\sin_{p'}(\pi_{p'}(\frac{1}{2} - x))) < 0$. Substituting the identity from Lemma 1-a, yields

$$\begin{aligned}\frac{d}{dp} \cos_p(\pi_p x) &= \frac{d}{dp} \left[\sin_{p'} \left(\pi_{p'} \left(\frac{1}{2} - x \right) \right) \right]^{\frac{1}{p-1}} \\ &= \left[-\frac{\ln(\sin_{p'}(\pi_{p'}(\frac{1}{2} - x)))}{(p-1)^2} + \frac{\frac{d}{dp} [\sin_{p'}(\pi_{p'}(\frac{1}{2} - x))]}{(p-1) \sin_{p'}(\pi_{p'}(\frac{1}{2} - x))} \right] \cos_p(\pi_p x) > 0.\end{aligned}$$

This implies “b”. \square

2.1 The case $1 < p < 2$

For $1 < p < 2$, let $u_p : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ be given by

$$u_p(x) := \cos'_p(\pi_p x) = -\sin_p(\pi_p x)^{p-1} \cos_p(\pi_p x)^{2-p}.$$

This function will simplify the notation when we determine estimates for the Fourier coefficients of the p -cosine functions in Section 3.1. Here and everywhere below we write

$$(6) \quad c_p := (p-1)^{\frac{p-1}{p}} (2-p)^{\frac{2-p}{p}}.$$

Lemma 3. *Let $1 < p < 2$. Then*

- a. $u_p(x) \leq 0$ for all $x \in [0, \frac{1}{2}]$
- b. $u_p(x) = 0$ if and only if $x = 0$ or $x = \frac{1}{2}$
- c. $u_p(x) = -c_p$ for $x \in [0, \frac{1}{2}]$ if and only if $x = m_p \in (0, \frac{1}{2})$, where m_p is the unique point such that $\cos_p(\pi_p m_p)^p = 2 - p$
- d. $u_p : [0, m_p] \longrightarrow [-c_p, 0]$ is decreasing
- e. $u_p : [m_p, \frac{1}{2}] \longrightarrow [-c_p, 0]$ is increasing
- f. $\min_{x \in [0, \frac{1}{2}]} u_p(x) = -c_p$.

Proof. Since $\sin_p(\pi_p x)$ and $\cos_p(\pi_p x)$ are non-negative over $[0, \frac{1}{2}]$, then “a” holds true. Since $\sin_p(\pi_p x)$ only vanishes at $x = 0$ and $\cos_p(\pi_p x)$ only vanishes at $x = \frac{1}{2}$ in this interval, then “b” holds true.

Lemma 1-c gives

$$u'_p(x) = \pi_p \sin_p(\pi_p x)^{p-2} \cos_p(\pi_p x)^{3-2p} [2 - p - \cos_p(\pi_p x)^p].$$

Neither \sin_p nor \cos_p vanish in $(0, \frac{1}{2})$. On the other hand, $\cos_p(0) = 1 > 2 - p$, $\cos_p(\frac{\pi_p}{2}) = 0 < 2 - p$ and $\cos_p(\pi_p x)^p$ is decreasing for $x \in (0, \frac{1}{2})$. Then the term $\cos_p(\pi_p x)^p + p - 2$ indeed vanishes at the unique point $m_p \in (0, \frac{1}{2})$ as stated in “c”.

At m_p ,

$$\begin{aligned} u_p(m_p) &= -\sin_p(\pi_p m_p)^{p-1} \cos_p(\pi_p m_p)^{2-p} \\ &= -(1 - \cos_p(\pi_p m_p)^p)^{\frac{p-1}{p}} \cos_p(\pi_p m_p)^{2-p} = -c_p. \end{aligned}$$

Hence, the proof of “d” and “e”, and thus of “f”, is achieved as follows. Just observe that in the expression for $u'_p(x)$ above, $\cos_p(\pi_p x)^p > 2 - p$ for $x \in [0, m_p)$ and $\cos_p(\pi_p x)^p < 2 - p$ for $x \in (m_p, \frac{1}{2})$, because $\cos_p(\pi_p x)$ is decreasing in $x \in (0, \frac{1}{2})$. \square

According to parts “d” and “e” of Lemma 3, the function u_p is invertible, when restricted to the segments $[0, m_p]$ and $[m_p, \frac{1}{2}]$. We denote the inverses by $w_{1,p} : [-c_p, 0] \longrightarrow [0, m_p]$ and $w_{2,p} : [-c_p, 0] \longrightarrow [m_p, \frac{1}{2}]$, respectively, so that

$$u_p(w_{k,p}(x)) = x \quad \forall x \in [-c_p, 0] \quad k = 1, 2.$$

2.2 The case $p > 2$

For $p > 2$, let $v_p : (0, \frac{1}{2}] \rightarrow [0, \infty)$ be given by

$$v_p(x) := (p' - 1) \sin_{p'}(\pi_{p'} x)^{p'-2} \cos_{p'}(\pi_{p'} x).$$

Let us summarise various properties of this function, which will be employed in Section 3.2.

Lemma 4. *Let $p > 2$. Then*

- a. v_p is decreasing in $(0, \frac{1}{2}]$
- b. $\lim_{x \rightarrow 0^+} x v_p(x) = 0$
- c. $\lim_{x \rightarrow 0^+} v_p(x) = +\infty$ and $v_p(\frac{1}{2}) = 0$
- d. $\lim_{x \rightarrow 0^+} v_p'(x) = -\infty$ and $v_p'(\frac{1}{2}) = 0$.

Proof. For $p > 2$, $p' \in (1, 2)$ and so $p' - 2 < 0$. Since, $\sin_{p'}(\pi_{p'} x)$ is increasing and $\cos_{p'}(\pi_{p'} x)$ is decreasing in $x \in (0, \frac{1}{2})$, then “a” holds true.

Let us show “b”. L’Hôpital’s Rule gives

$$\lim_{x \rightarrow 0^+} \frac{x}{[\sin_{p'}(\pi_{p'} x)]^{2-p'}} = \lim_{x \rightarrow 0^+} \frac{[\sin_{p'}(\pi_{p'} x)]^{p'-1}}{(2-p')\pi_{p'} \cos_{p'}(\pi_{p'} x)} = 0.$$

Then,

$$\lim_{x \rightarrow 0^+} x v_p(x) = \lim_{x \rightarrow 0^+} (p' - 1) \frac{x \cos_{p'}(\pi_{p'} x)}{[\sin_{p'}(\pi_{p'} x)]^{2-p'}} = 0,$$

as claimed in “b”.

Both statements “c” and “d” follow directly from (5) and the expression

$$v_p'(x) = (p' - 1)\pi_{p'} \sin_{p'}(\pi_{p'} x)^{p'-3} \cos_{p'}(\pi_{p'} x)^{2-p'} \left[(p' - 1) \cos_{p'}(\pi_{p'} x)^{p'} - 1 \right].$$

□

According to this lemma, there exists a function $z_p : [0, \infty) \rightarrow (0, \frac{1}{2}]$ such that z_p is the inverse function of v_p . This inverse function has the following characteristics.

- a. z_p is decreasing in $[0, \infty)$
- b. $z_p(0) = \frac{1}{2}$ and $\lim_{x \rightarrow \infty} z_p(x) = 0$
- c. $\lim_{x \rightarrow 0^+} z_p'(x) = +\infty$ and $\lim_{x \rightarrow \infty} z_p'(x) = 0$.

3 The Fourier coefficients of the p -cosine functions

Let

$$a_j(p) \equiv a_j := 2 \int_0^1 \sin_p(\pi_p x) \sin(j\pi x) dx \quad \forall j \in \mathbb{N}$$

be the Fourier sine coefficients of $\sin_p(\pi_p x)$. Let

$$b_j(p) \equiv b_j := 2 \int_0^1 \cos_p(\pi_p x) \cos(j\pi x) dx \quad \forall j \in \mathbb{N} \cup \{0\}$$

be the Fourier cosine coefficients of $\cos_p(\pi_p x)$. Since \sin_p is an odd function and \cos_p is an even function, $a_j = b_j = 0$ for all $j \equiv_2 0$.

Lemma 5. For $j \in \mathbb{N}$,

$$b_j(p) = \frac{j\pi}{\pi_p} a_j(p).$$

Proof. Let $j \equiv_2 1$. Integration by parts alongside with the fact that $\cos_p(\pi_p x)$ and $\cos(j\pi x)$ are odd with respect to $\frac{1}{2}$, yield

$$\begin{aligned} b_j &= 2 \int_0^1 \cos_p(\pi_p x) \cos(j\pi x) dx = 4 \int_0^{\frac{1}{2}} \cos_p(\pi_p x) \cos(j\pi x) dx \\ &= \frac{4}{\pi_p} \cos(j\pi x) \sin_p(\pi_p x) \Big|_0^{\frac{1}{2}} + \frac{4j\pi}{\pi_p} \int_0^{\frac{1}{2}} \sin_p(\pi_p x) \sin(j\pi x) dx \\ &= \frac{j\pi}{\pi_p} a_j. \end{aligned}$$

□

We now find estimates on $|b_j(p)|$ in terms of the parameter $p > 1$.

3.1 The case $1 < p < 2$

Lemma 6. For $1 < p < 2$, let $c_p > 0$ be given by (6). Then

$$|b_j(p)| < \frac{8\pi_p}{j^2\pi^2} c_p \quad \forall j \geq 1.$$

Proof. Integrate by parts twice to get

$$\begin{aligned} b_j &= 4 \int_0^{\frac{1}{2}} \cos_p(\pi_p x) \cos(j\pi x) dx \\ &= \frac{4}{j\pi} \cos_p(\pi_p x) \sin(j\pi x) \Big|_0^{\frac{1}{2}} - \frac{4\pi_p}{j\pi} \int_0^{\frac{1}{2}} \cos'_p(\pi_p x) \sin(j\pi x) dx \\ &= -\frac{4\pi_p}{j\pi} \int_0^{\frac{1}{2}} \cos'_p(\pi_p x) \sin(j\pi x) dx \\ &= \frac{4\pi_p}{j^2\pi^2} \cos'_p(\pi_p x) \cos(j\pi x) \Big|_0^{\frac{1}{2}} - \frac{4\pi_p}{j^2\pi^2} \int_0^{\frac{1}{2}} \frac{d}{dx} [\cos'_p(\pi_p x)] \cos(j\pi x) dx. \end{aligned}$$

From the identities in Lemma 3-b, it follows that the boundary term in the fourth equality always vanishes. Thus,

$$\begin{aligned}
b_j &= -\frac{4\pi_p}{j^2\pi^2} \int_0^{\frac{1}{2}} u'_p(x) \cos(j\pi x) dx \\
&= -\frac{4\pi_p}{j^2\pi^2} \left(\int_0^{m_p} + \int_{m_p}^{\frac{1}{2}} u'_p(x) \cos(j\pi x) dx \right) \\
&= -\frac{4\pi_p}{j^2\pi^2} \left(\int_0^{-c_p} \cos(j\pi w_{1,p}(s)) ds + \int_{-c_p}^0 \cos(j\pi w_{2,p}(s)) ds \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
|b_j| &\leq \frac{4\pi_p}{j^2\pi^2} \left[\int_{-c_p}^0 |\cos(j\pi w_{1,p}(s))| ds + \int_{-c_p}^0 |\cos(j\pi w_{2,p}(s))| ds \right] \\
&< \frac{8\pi_p}{j^2\pi^2} c_p,
\end{aligned}$$

because the functions inside the integrals are not constants identically equal to 1. \square

3.2 The case $p > 2$

Let $p > 2$. According to Lemma 1-a,

$$b_j(p) = 4 \int_0^{\frac{1}{2}} \sin_{p'} \left(\pi_{p'} \left(\frac{1}{2} - x \right) \right)^{\frac{1}{p-1}} \cos(j\pi x) dx.$$

Since $\cos(j\pi(\frac{1}{2} - t)) = (-1)^{\frac{j-1}{2}} \sin(j\pi t)$ for $j \equiv_2 1$, changing variables to $t = \frac{1}{2} - x$ gives

$$b_j = (-1)^{\frac{j-1}{2}} 4 \int_0^{\frac{1}{2}} \sin_{p'}(\pi_{p'} t)^{\frac{1}{p-1}} \sin(j\pi t) dt.$$

By virtue of Lemma 4 and integration by parts twice, then

$$\begin{aligned}
b_j &= (-1)^{\frac{j-1}{2}} \frac{4\pi_{p'}}{j\pi} \int_0^{\frac{1}{2}} v_p(t) \cos(j\pi t) dt \\
&= (-1)^{\frac{j-1}{2}} \frac{4\pi_{p'}}{j\pi} \left[\frac{1}{j\pi} v_p(t) \sin(j\pi t) \Big|_0^{\frac{1}{2}} - \frac{1}{j\pi} \int_0^{\frac{1}{2}} v'_p(t) \sin(j\pi t) dt \right] \\
&= (-1)^{\frac{j+1}{2}} \frac{4\pi_{p'}}{j^2\pi^2} \int_0^{\frac{1}{2}} v'_p(t) \sin(j\pi t) dt \\
(7) \quad &= (-1)^{\frac{j+3}{2}} \frac{4\pi_{p'}}{j^2\pi^2} \int_0^\infty \sin(j\pi z_p(y)) dy.
\end{aligned}$$

Lemma 7. *Let $p > 2$. Then*

$$|b_j(p)| < \frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2) \right] j^{-p'} \quad \forall j \geq 3.$$

Proof. Since $p > 2$, then $1 < p' < 2$. Let $r = p' - 1$. In view of Lemma 2, we have

$$v_p(t) \leq r [\sin_{p'}(\pi_{p'} t)]^{r-1} \leq r [\sin(\pi t)]^{r-1}$$

and so

$$(8) \quad z_p(y) \leq \frac{1}{\pi} \arcsin \left[\left(\frac{y}{r} \right)^{\frac{1}{r-1}} \right] =: r_p(y) \quad \forall y \in [r, \infty).$$

Set

$$\eta(j) := r \sin \left(\frac{\pi}{2j} \right)^{r-1}.$$

Then,

$$r_p(\eta(j)) = \frac{1}{2j} < \frac{1}{2}.$$

Here we use the requirement $j \geq 3$, in order to make sure that the arc-sine does not change branches.

Set

$$J_1 = \int_0^{\eta(j)} dx = \eta(j)$$

and

$$J_2 = \int_{\eta(j)}^{\infty} \sin(j\pi r_p(y)) dy.$$

Then, (7) yields

$$|b_j| \leq \frac{4\pi_{p'}}{j^2\pi^2} (J_1 + J_2).$$

Here J_2 is guaranteed to be on the right hand side, because

$$0 < j\pi z_p(y) \leq j\pi z_p(\eta(j)) \leq j\pi r_p(\eta(j)) = \frac{\pi}{2},$$

so that $0 < \sin(j\pi z_p(y)) \leq \sin(j\pi r_p(y))$ for $y \in [\eta(j), \infty)$.

Let us estimate an upper bound for J_2 . Changing variables to

$$t = j\pi r_p(y) \iff y = r \sin \left(\frac{t}{j} \right)^{r-1}$$

gives

$$\begin{aligned} J_2 &= \int_0^{\frac{\pi}{2}} \frac{r(1-r)}{j} \sin \left(\frac{t}{j} \right)^{r-2} \cos \left(\frac{t}{j} \right) \sin(t) dt \\ &= r(1-r) \int_0^{\frac{\pi}{2}} \sin \left(\frac{t}{j} \right)^{r-1} \left[\frac{\frac{t}{j}}{\sin \left(\frac{t}{j} \right)} \right] \left(\frac{\sin t}{t} \right) \cos \left(\frac{t}{j} \right) dt. \end{aligned}$$

Note that,

$$(9) \quad \max_{0 < \theta \leq \frac{\pi}{2}} \frac{\theta}{\sin \theta} = \frac{\pi}{2}, \quad \max_{0 < \theta \leq \frac{\pi}{2}} \frac{\sin \theta}{\theta} = 1$$

and

$$0 < t < j\pi \mathbf{r}_p(\eta(j)) = \frac{\pi}{2}.$$

Here we are using once again the fact that $j \geq 3$. Then

$$J_2 < \frac{\pi}{2} r(1-r) \int_0^{\frac{\pi}{2}} \sin\left(\frac{t}{j}\right)^{r-1} \cos\left(\frac{t}{j}\right) dt.$$

Changing variables to

$$\tau = \sin\left(\frac{t}{j}\right),$$

yields

$$J_2 < \frac{j\pi}{2} r(1-r) \int_0^{\sin \frac{\pi}{2j}} \tau^{r-1} d\tau = \frac{j\pi}{2} (1-r) \sin\left(\frac{\pi}{2j}\right)^r.$$

Then

$$|b_j| < \frac{2\pi_{p'}}{j^2\pi^2} \left[2 + \frac{j\pi(1-r)}{r} \sin\left(\frac{\pi}{2j}\right) \right] \eta(j).$$

According to (9), we get

$$\eta(j) \leq rj^{1-r}$$

and

$$(10) \quad |b_j| < \frac{2\pi_{p'}r}{j^2\pi^2} \left[2 + \frac{j\pi(1-r)}{r} \frac{\pi}{2j} \right] j^{1-r}.$$

Simplifying the expression on the right hand side, ensures the validity of the lemma. \square

4 The change of coordinates map

We now derive various properties of the change of coordinates maps that take the 2-cosine functions into the p -cosine functions. Most of the material in this section can also be found in [2], [5], [7] and [4]. We keep a self-contained presentation here by including details of the main arguments.

Given any $g \in L_s$, denote the even extension of g with respect to 1 by

$$\tilde{g}(x) = \begin{cases} g(x) & x \in [0, 1] \\ g(2-x) & x \in (1, 2]. \end{cases}$$

A 2-periodic extension of g to the whole of \mathbb{R} is then written as

$$g^*(x) = \tilde{g}(x - 2 \left\lfloor \frac{x}{2} \right\rfloor).$$

The floor function $\lfloor y \rfloor \in \mathbb{Z}$ is the unique integer such that $y - \lfloor y \rfloor \in [0, 1)$. For any $n \in \mathbb{N}$, let

$$M_n g(x) := g^*(nx).$$

Lemma 8. *The operators $M_n : L_s \rightarrow L_s$ are linear isometries.*

Proof. Indeed,

$$\begin{aligned} \|M_n g\|_{L_s}^s &= \int_0^1 |M_n g(x)|^s dx = \int_0^1 |g^*(nx)|^s dx = \int_0^1 |\tilde{g}(nx - 2 \left\lfloor \frac{nx}{2} \right\rfloor)|^s dx \\ &= \frac{1}{n} \int_0^n |\tilde{g}(y - 2 \left\lfloor \frac{y}{2} \right\rfloor)|^s dy = \frac{1}{n} \sum_{l=0}^{n-1} \int_l^{l+1} |\tilde{g}(y - 2 \left\lfloor \frac{y}{2} \right\rfloor)|^s dy \\ &= \frac{1}{n} \left[\sum_{\substack{l=0 \\ l \equiv 0}}^{n-1} \int_l^{l+1} |\tilde{g}(y - 2 \left\lfloor \frac{y}{2} \right\rfloor)|^s dy + \sum_{\substack{l=1 \\ l \equiv 1}}^{n-1} \int_l^{l+1} |\tilde{g}(y - 2 \left\lfloor \frac{y}{2} \right\rfloor)|^s dy \right]. \end{aligned}$$

Changing variables to $w = y - l$ for $l \equiv 0$ and $z = y - (l - 1)$ for $l \equiv 1$, gives

$$\left\lfloor \frac{y}{2} \right\rfloor = \begin{cases} \frac{l}{2} & \text{whenever } l \equiv 0 \\ \frac{l-1}{2} & \text{whenever } l \equiv 1. \end{cases}$$

Hence,

$$\|M_n g\|_{L_s}^s = \frac{1}{n} \left[\sum_{\substack{l=0 \\ l \equiv 0}}^{n-1} \int_0^1 |g(w)|^s dw + \sum_{\substack{l=1 \\ l \equiv 1}}^{n-1} \int_1^2 |\tilde{g}(z)|^s dz \right].$$

Another change of variables $z = 2 - w$, then yields

$$\|M_n g\|_{L_s}^s = \frac{1}{n} \left[n \int_0^1 |g(w)|^s dw \right] = \|g\|_{L_s}^s$$

as claimed. \square

Let $e_n(x) := \cos(n\pi x)$. If

$$g = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) e_j \in L_s$$

where

$$\widehat{g}(k) := 2 \int_0^1 g(x) e_k(x) dx \quad \forall k \in \mathbb{N} \cup \{0\}$$

are the corresponding cosine Fourier coefficients, then

$$M_n g = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) M_n e_j = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) e_{nj} \in L_s.$$

Now, let $f_n(x) := \cos_p(n\pi_p x)$. Note that $e_0(x) = f_0(x) = 1$ for all $x \in \mathbb{R}$. Suitable linear extensions of the map $A : e_n \mapsto f_n$ are the changes of coordinates between $\{e_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$. Our next goal is to find a canonical decomposition for A in terms of M_n and the Fourier coefficients $b_n(p)$. After that, we show that these are bounded operators of the Banach spaces L_s for all $s > 1$.

Proposition 1. *For all $p > 1$,*

$$\sum_{j=1}^{\infty} |b_j(p)| < \infty.$$

Proof. This is a direct consequence of lemmas 6 and 7. See (14) and (23) below. \square

In the notation of Section 3, we have $\widehat{f}_1(k) = b_k(p)$ for all $k \in \mathbb{N} \cup \{0\}$. Recall that $b_k = 0$ for $k \equiv_2 0$. Since any of the functions $f_n(x)$ is continuous, then they all have a Fourier cosine expansion

$$f_n(x) = \frac{1}{2} \widehat{f}_n(0) e_0(x) + \sum_{k=1}^{\infty} \widehat{f}_n(k) e_k(x)$$

which is both pointwise convergent for all $x \in [0, 1]$ and also convergent in the norm of L_s for all $s > 1$. Then, for all $n > 1$,

$$\begin{aligned} \widehat{f}_n(k) &= 2 \int_0^1 f_1(nx) \cos(k\pi x) dx \\ &= 2 \int_0^1 \left(\sum_{m=1}^{\infty} \widehat{f}_1(m) \cos(m\pi nx) \right) \cos(k\pi x) dx \\ &= 2 \sum_{m=1}^{\infty} \widehat{f}_1(m) \int_0^1 \cos(mn\pi x) \cos(k\pi x) dx \\ &= \begin{cases} b_m(p) & \text{for } mn = k, \ m \equiv_2 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here we can exchange the infinite summation with the integral sign, due to the pointwise convergence of the series, Proposition 1 and the Dominated Convergence theorem.

Let

$$(11) \quad A := \sum_{j=1}^{\infty} b_j(p) M_j.$$

By virtue of Proposition 1, Lemma 8 and the triangle inequality, it follows that the expression (11) is convergent in the operator norm of L_s and that $A : L_s \rightarrow L_s$ is a bounded linear operator such that

$$\|A\|_{L_s \rightarrow L_s} \leq \sum_{j=1}^{\infty} |b_j| \|M_j\|_{L_s \rightarrow L_s} = \sum_{j=1}^{\infty} |b_j|.$$

Moreover,

$$Ae_0 = \sum_{j=1}^{\infty} b_j M_j e_0 = \sum_{j=1}^{\infty} b_j e_0 = \sum_{j=1}^{\infty} b_j e_j(0) = \cos_p(\pi_p 0) = 1 = f_0$$

and

$$Ae_n = \sum_{j=1}^{\infty} b_j M_j e_n = \sum_{j=1}^{\infty} \hat{f}_1(j) e_{nj} = \sum_{k=1}^{\infty} \hat{f}_n(k) e_k = f_n \quad \forall n \in \mathbb{N}.$$

These are the change of basis maps between $\{e_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$.

The operator A is an homeomorphism of L_s if and only if the family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ is a Schauder basis of L_s , cf. [12] or [15]. Then we have the following criterion, which is a consequence of [13, Theorem IV-1.16],

$$(12) \quad \sum_{\substack{j=3 \\ j=2^1}}^{\infty} |b_j(p)| < |b_1(p)| \quad \Rightarrow \quad \begin{cases} \{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty} \text{ is a Schauder} \\ \text{basis of } L_s \text{ for all } s > 1. \end{cases}$$

We employ this criterion below in order to determine the basis thresholds for the family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ claimed in Theorem 1.

5 Proof of Theorem 1

The proof is separated into two cases.

5.1 The case $1 < p < 2$

Recall the expression for c_p given in (6) and consider the identity

$$(13) \quad \pi_p^2 c_p = \frac{\pi^3}{\pi^2 - 8}.$$

Lemma 9. *There exists $1 < p_0 < 2$ such that (13) holds true for $p = p_0$. Moreover,*

$$\pi_p^2 c_p < \frac{\pi^3}{\pi^2 - 8} \quad \forall p \in (p_0, 2).$$

Proof. It will be enough to prove that $\pi_p^2 c_p$ is a convex function of the parameter p for all $1 < p < 2$. Indeed, since

$$\lim_{p \rightarrow 1^+} \pi_p^2 c_p = \infty \quad \text{and} \quad \lim_{p \rightarrow 2^-} \pi_p^2 c_p = \pi^2 < \frac{\pi^3}{\pi^2 - 8},$$

both statements will immediately follow from this property.

Firstly note that

$$\frac{d}{dp} \ln(p-1)^{\frac{p-1}{p}} = \frac{1}{p^2} \ln(p-1) + \frac{1}{p}$$

and

$$\frac{d^2}{dp^2} \ln(p-1)^{\frac{p-1}{p}} = \frac{2-p}{p^2(p-1)} - 2 \frac{\ln(p-1)}{p^3} > 0.$$

Then $\ln(p-1)^{\frac{p-1}{p}}$ is convex for $1 < p < 2$.

Similarly, we have

$$\frac{d}{dp} \ln(2-p)^{\frac{2-p}{p}} = \frac{-2}{p^2} \ln(2-p) - \frac{1}{p}$$

and

$$\frac{d^2}{dp^2} \ln(2-p)^{\frac{2-p}{p}} = \frac{4-p}{p^2(2-p)} + 4 \frac{\ln(2-p)}{p^3} > 0.$$

Then, also $\ln(2-p)^{\frac{2-p}{p}}$ is convex for $1 < p < 2$.

Furthermore,

$$\frac{d}{dp} [\ln \pi_p] = \frac{\pi \cot(\frac{\pi}{p})}{p^2} - \frac{1}{p}$$

and

$$\frac{d^2}{dp^2} \ln \pi_p = \frac{(p^2 + \pi^2)}{p^4} - \frac{2\pi}{p^3} \cot\left(\frac{\pi}{p}\right) + \frac{\pi^2}{p^4} \cot^2\left(\frac{\pi}{p}\right) > 0.$$

The latter is a consequence of the fact that $\cos \frac{\pi}{p} < 0$ and $\sin \frac{\pi}{p} > 0$. Hence, also $\ln \pi_p^2$ is convex for $1 < p < 2$.

The convexity of the logarithm of each one of the multiplying terms in the expression for $\pi_p^2 c_p$, implies that $\ln \pi_p^2 c_p$ is convex for $1 < p < 2$. This ensures that indeed $\pi_p^2 c_p$ is convex in the same segment and the validity of the statement is ensured. \square

Corollary 1. *Let $1 < p_0 < 2$ be such that (13) holds true for $p = p_0$. The family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ is a Schauder basis of L_s for all $s > 1$ and $p_0 \leq p \leq 2$.*

Proof. According to Lemma 6,

$$(14) \quad \sum_{\substack{j=3 \\ j \equiv 21}}^{\infty} |b_j(p)| < \frac{8\pi_p c_p}{\pi^2} \sum_{\substack{j=3 \\ j \equiv 21}}^{\infty} \frac{1}{j^2} = \frac{\pi_p^2 c_p (\pi^2 - 8)}{\pi^2 \pi_p}.$$

On the other hand, in view of Lemma 5 and Lemma 2-a, we have

$$\begin{aligned} b_1(p) &= \frac{\pi}{\pi_p} a_1 = \frac{4\pi}{\pi_p} \int_0^{\frac{1}{2}} \sin_p(\pi_p x) \sin(\pi x) dx \\ &\geq \frac{4\pi}{\pi_p} \int_0^{\frac{1}{2}} \sin(\pi x)^2 dx = \frac{\pi}{\pi_p}. \end{aligned}$$

Then, Lemma 9 yields

$$\sum_{\substack{j=3 \\ j \equiv 21}}^{\infty} |b_j(p)| < b_1(p)$$

for all $p \in [p_0, 2)$. By virtue of (12) the claimed conclusion follows. \square

Since

$$\pi_{\frac{4}{3}}^2 c_{\frac{4}{3}} = \frac{\pi^2 3^{\frac{5}{4}} \sqrt{2}}{2} > \frac{\pi^3}{\pi^2 - 8}$$

and

$$\pi_{\frac{3}{2}}^2 c_{\frac{3}{2}} = \frac{64\pi^2}{27\sqrt[3]{4}} < \frac{\pi^3}{\pi^2 - 8},$$

then $\frac{4}{3} < p_0 < \frac{3}{2}$. This settles the proof of Theorem 1 for $1 < p < 2$.

Remark 1. *An implementation of the Newton method gives $p_0 \approx 1.458801$ as an approximated solution of (13) with all digits correct.*

5.2 Case $p > 2$

Recall the following identities involving the Riemann Zeta function [11, 3.411, 9.522 & 9.524],

$$(15) \quad \zeta(q) = \frac{1}{\Gamma(q)} \int_0^{\infty} \frac{t^{q-1}}{e^t - 1} dt \quad \operatorname{Re}(q) > 1,$$

$$(16) \quad \sum_{\substack{j=1 \\ j \neq 20}}^{\infty} \frac{1}{j^q} = \left(1 - \frac{1}{2^q}\right) \zeta(q)$$

and

$$(17) \quad \frac{\zeta'(q)}{\zeta(q)} = - \sum_{k=1}^{\infty} \frac{\Delta(k)}{k^q}$$

where

$$\Delta(k) = \begin{cases} \ln(r) & \text{if } k = r^m \text{ for some } r \text{ prime and } m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 10. *Let*

$$t_0 = \frac{2(e^2 - 3e + 1)}{(e^2 - 2e - 1)}.$$

Then

$$(18) \quad \zeta\left(\frac{3}{2}\right) < \frac{2}{\sqrt{\pi}} \left(2\sqrt{2} \arctan \frac{1}{\sqrt{2}} + \frac{\pi^2}{6} + \frac{t_0^2}{4} - \frac{(t_0 - 1)^2}{2(e - 1)^2} - \frac{t_0(e - 2) + 1}{e - 1} \right).$$

Proof. Since $\Gamma(1 + \frac{1}{2}) = \frac{\sqrt{\pi}}{2} 1!! = \frac{\sqrt{\pi}}{2}$, the representation (15) gives

$$\begin{aligned} \zeta\left(\frac{3}{2}\right) &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{t^{1/2}}{e^t - 1} dt \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^1 + \int_1^{\infty} \frac{t^{1/2}}{e^t - 1} dt \right) = \frac{2}{\sqrt{\pi}} (J_1 + J_2). \end{aligned}$$

We estimate separately upper bounds for J_1 and J_2 .

The change of variables $t = u^2$, yields

$$\begin{aligned} J_1 &= \int_0^1 \frac{t^{1/2}}{e^t - 1} dt < \int_0^1 \frac{t^{1/2}}{t + \frac{t^2}{2}} dt \\ &= \int_0^1 \frac{2u^2}{u^2 + \frac{u^4}{2}} du = 2\sqrt{2} \arctan \frac{1}{\sqrt{2}}. \end{aligned}$$

On the other hand, we know that $\zeta(2) = \int_0^{\infty} \frac{t}{e^t - 1} dt = \frac{\pi^2}{6}$, so

$$J_2 \leq \int_1^{\infty} \frac{t}{e^t - 1} dt = \frac{\pi^2}{6} - \int_0^1 \frac{t}{e^t - 1} dt.$$

We find lower bound for the integral on the right hand side, by interpolating the curve $c(t) = \frac{t}{e^t - 1}$ at two points, $t = 0$ and $t = 1$. Firstly observe that $c(t) \rightarrow 1$ as $t \rightarrow 0$, $c(t)$ is decreasing and $c''(t) \geq 0$ for $t \in [0, 1]$. Let t_0 be as in the hypothesis and let

$$\tilde{c}(t) = \begin{cases} 1 - \frac{1}{2}t & 0 \leq t \leq t_0 \\ \frac{1}{(e-1)^2}(1-t) + \frac{1}{e-1} & t_0 \leq t \leq 1 \end{cases}$$

be the piecewise linear interpolant of $c(t)$ in the two segments $[0, t_0]$ and $[t_0, 1]$, which is continuous at t_0 . Note that $\tilde{c}(t)$ and $c(t)$ are tangent at $t = 0$ and $t = 1$. Then

$$c(t) \geq \tilde{c}(t) \quad \forall t \in [0, 1].$$

Hence

$$\begin{aligned} \int_0^1 c(t) dt &\geq \int_0^{t_0} \left(1 - \frac{1}{2}t\right) dt + \int_{t_0}^1 \left(\frac{1}{(e-1)^2}(1-t) + \frac{1}{e-1}\right) dt \\ &= -\frac{t_0^2}{4} + \frac{(t_0-1)^2}{2(e-1)^2} + \frac{t_0(e-2)+1}{e-1}. \end{aligned}$$

Thus

$$J_2 \leq \frac{\pi^2}{6} + \frac{t_0^2}{4} - \frac{(t_0-1)^2}{2(e-1)^2} - \frac{t_0(e-2)+1}{e-1}.$$

Alongside with the upper bound above for J_1 , this ensures the validity of the claimed statement. \square

Now, consider the equation

$$(19) \quad \frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1\right] = \frac{8}{\pi\pi_p}.$$

Lemma 11. *There exists $p_1 \in (\frac{11}{5}, 3)$ such that (19) holds true for $p = p_1$. Moreover,*

$$\frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1\right] < \frac{8}{\pi\pi_p} \quad \forall p \in [2, p_1].$$

Proof. From (4) it follows that the identity (19) reduces to

$$(20) \quad \frac{\pi}{p^2 \sin(\frac{\pi}{p})^2} \left(2 + \frac{\pi^2}{2}(p-2)\right) \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1\right] = 1.$$

Denote by $h(p)$ the left hand side of (20). Then $h : (1, \infty) \rightarrow \mathbb{R}$ is continuous and

$$h(2) = \frac{\pi}{2} \left(\frac{\pi^2}{8} - 1\right) < 1.$$

Since

$$\zeta\left(\frac{3}{2}\right) > 1 + \frac{\sqrt{2}}{4} + \sqrt{3} \sum_{k=3}^{\infty} \frac{1}{k^2} = \frac{4+\sqrt{2}}{4} + \sqrt{3} \left(\frac{\pi^2}{6} - \frac{5}{4}\right),$$

we get

$$\begin{aligned} h(3) &= \frac{\pi}{9 \sin(\frac{\pi}{3})^2} \left[2 + \frac{\pi^2}{2}\right] \left[\left(1 - \frac{1}{2^{\frac{3}{2}}}\right) \zeta\left(\frac{3}{2}\right) - 1\right] \\ &> \frac{\pi}{9 \sin(\frac{\pi}{3})^2} \left[2 + \frac{\pi^2}{2}\right] \left[\left(1 - \frac{1}{2^{\frac{3}{2}}}\right) \left(\frac{4+\sqrt{2}}{4} + \sqrt{3} \left(\frac{\pi^2}{6} - \frac{5}{4}\right)\right) - 1\right] \\ &> 1. \end{aligned}$$

Hence, there exists $p_1 \in (2, 3)$ such that $h(p_1) = 1$.

The derivative

$$\frac{d}{dq} \left[\left(1 - \frac{1}{2^q} \right) \zeta(q) \right] = \frac{\ln(2)}{2^q} \zeta(q) + \left(1 - \frac{1}{2^q} \right) \zeta'(q)$$

is negative for any $q \in (1, 2)$. Indeed the identity (17) gives

$$\begin{aligned} \frac{\zeta'(q)}{\zeta(q)} &< -\frac{\ln(2)}{2^q} - \frac{\ln(3)}{3^q} - \frac{\ln(2)}{4^q} \\ &< -\ln(2) \left[\frac{1}{2^q} + \frac{1}{3^q} + \frac{1}{4^q} \right] < \frac{\ln(2)}{1 - 2^q}, \end{aligned}$$

so that

$$\frac{d}{dq} \left[\left(1 - \frac{1}{2^q} \right) \zeta(q) \right] = \zeta(q) \left[\frac{\ln(2)}{2^q} + \frac{2^q - 1}{2^q} \frac{\zeta'(q)}{\zeta(q)} \right] < 0.$$

Since p' and $\sin\left(\frac{\pi}{p}\right)$ are decreasing functions of $p > 2$, then

$$\frac{\pi}{\sin\left(\frac{\pi}{p}\right)^2} \left[\left(1 - \frac{1}{2^{p'}} \right) \zeta(p') - 1 \right]$$

is an increasing function of $p > 2$.

As

$$\frac{d}{dp} \left[\frac{1}{p^2} \left(2 + \frac{\pi^2}{2}(p - 2) \right) \right] = \frac{1}{p^3} \left(-\frac{\pi^2}{2}p + 2\pi^2 - 4 \right) > 0 \quad \forall p \in [2, 3],$$

then $h(p)$ is increasing for $p \in [2, 3]$ and so indeed

$$h(p) < h(p_1) = 1 \quad \forall p \in [2, p_1).$$

Let us now show that $p_1 > \frac{11}{5}$. Let c_1 denote the right hand side of the estimate (18) in Lemma 10. Since $\zeta(q)$ is convex in the segment $[\frac{3}{2}, 2]$, then

$$\zeta(q) \leq \left(\frac{\pi^2}{3} - 2c_1 \right) (q - 2) + \frac{\pi^2}{6}.$$

That is, the straight line joining the points $(\frac{3}{2}, c_1)$ and $(2, \frac{\pi^2}{6})$ is above the curve $\zeta(q)$ for all $q \in [\frac{3}{2}, 2]$. Then

$$(21) \quad \zeta\left(\frac{11}{6}\right) \leq \frac{\pi^2}{9} + \frac{c_1}{3}.$$

Note that for $p = \frac{11}{5}$, $p' = \frac{11}{6}$. Now, $\sin(\pi y)$ is concave for $y \in [\frac{5}{12}, \frac{1}{2}]$. Then it is above the straight line joining the points $(\frac{5}{12}, \sin \frac{5\pi}{12})$ and $(\frac{1}{2}, 1)$. That is

$$\sin(\pi y) \geq \left(12 - 12 \sin \frac{5\pi}{12} \right) \left(y - \frac{1}{2} \right) + 1 \quad \forall y \in \left[\frac{5}{12}, \frac{1}{2} \right].$$

Then

$$(22) \quad \sin \frac{5\pi}{11} > \frac{\sqrt{6}}{22}(\sqrt{3} + 3) + \frac{5}{11}.$$

Denote by c_2 the right hand side of the latter inequality. From (21) and (22), it follows that

$$\begin{aligned} h\left(\frac{11}{5}\right) &= \frac{\pi}{\left(\frac{11}{5}\right)^2 \sin\left(\frac{5\pi}{11}\right)^2} \left[2 + \frac{\pi^2}{2} \left(\frac{11}{5} - 2\right)\right] \left[\left(1 - \frac{1}{2^{11/6}}\right) \zeta\left(\frac{11}{6}\right) - 1\right] \\ &< \frac{\pi}{\frac{121}{25} c_2^2} \left(2 + \frac{\pi^2}{10}\right) \left[\left(1 - \frac{1}{2^{11/6}}\right) \left(\frac{\pi^2}{9} + \frac{c_1}{3}\right) - 1\right] < 1. \end{aligned}$$

As $h(p)$ is increasing, then indeed $p_1 > \frac{11}{5}$. \square

Corollary 2. *Let $p_1 > 2$ be such that (19) holds true for $p = p_1$. The family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^\infty$ forms a Schauder basis of L_s for all $s > 1$ and $2 \leq p \leq p_1$.*

Proof. From Lemma 7 and (16), we have

$$(23) \quad \sum_{\substack{j=3 \\ j \equiv 21}}^\infty |b_j| < \frac{2\pi_{p'}}{\pi^2(p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1\right].$$

According to part “b” of Lemma 1, $\sin_p(\pi_p x)$ is strictly concave on $(0, \frac{1}{2})$. Then

$$\begin{aligned} a_1 &= 2 \int_0^1 \sin_p(\pi_p x) \sin(\pi x) dx = 4 \int_0^{\frac{1}{2}} \sin_p(\pi_p x) \sin(\pi x) dx \\ &> 4 \int_0^{\frac{1}{2}} (2x) \sin(\pi x) dx = \frac{8}{\pi^2}. \end{aligned}$$

Hence, in view of Lemma 5, we get

$$(24) \quad b_1 = \frac{\pi}{\pi_p} a_1 > \frac{8}{\pi \pi_p}.$$

From Lemma 11, it then follows that

$$\sum_{\substack{j=3 \\ j \equiv 21}}^\infty |b_j(p)| < b_1(p) \quad \forall p \in [2, p_1].$$

By virtue of (12) this implies the claimed conclusion. \square

Remark 2. *An approximation of the solution of (19) via the Newton Method gives $p_1 \approx 2.42865$ with all digits correct.*

6 Connections with other work

In this final section we describe various connections between the statements established above and those reported in the literature.

The p -exponential functions

Let

$$\exp_p(iy) = \cos_p(y) + i \sin_p(y) \quad \forall y \in \mathbb{R}.$$

By combining Theorem 1 with [2, Theorem 1] or [5, Theorem 4.5], it immediately follows that the family $\tilde{\mathcal{F}} = \{\exp_p(in\pi_p \cdot)\}_{n=-\infty}^{\infty}$ is a Schauder basis of the Banach space $L_s(-1, 1)$ for all $p \in [p_0, p_1]$.

Indeed, recall that every $f \in L^s(-1, 1)$ decomposes as $f = f_e + f_o$ for

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2},$$

the even and odd parts of f , respectively. The family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ comprises only even functions, the family $\{\sin_p(n\pi_p \cdot)\}_{n=1}^{\infty}$ comprises only odd functions and they are Schauder bases of the corresponding subspaces of $L_s(-1, 1)$ for $p \in [p_0, p_1]$. This implies that there exist two unique scalar sequences $(\alpha_k)_{k=0}^{\infty}$ and $(\beta_k)_{k=1}^{\infty}$, such that

$$f(\cdot) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos_p(k\pi_p \cdot) + i\beta_k \sin_p(k\pi_p \cdot)$$

in $L_s(-1, 1)$. In order to see this, one expands f_e in $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ and f_o in $\{\sin_p(n\pi_p \cdot)\}_{n=1}^{\infty}$, in the corresponding even and odd subspaces.

By letting $c_0 = \alpha_0$,

$$c_k = \frac{\alpha_k + \beta_k}{2} \quad \text{and} \quad c_{-k} = \frac{\alpha_k - \beta_k}{2} \quad \forall k \in \mathbb{N},$$

we get

$$f(\cdot) = \sum_{k=-\infty}^{\infty} c_k \exp_p(ik\pi_p \cdot)$$

in $L_s(-1, 1)$. Since there is a 1:1 correspondence between the scalar sequences via

$$\alpha_k = c_k + c_{-k} \quad \text{and} \quad \beta_k = c_k - c_{-k},$$

then in fact $(c_k)_{k=-\infty}^{\infty}$ is unique for the given f . Thus, $\tilde{\mathcal{F}}$ satisfies the definition of a Schauder basis for the Banach space $L_s(-1, 1)$.

The regularity of the p -sine functions

Let $r > 0$ and denote by $H^r \equiv H^r(0, 1)$ the (Hilbert) Sobolev space of order r . Let $1 < p < 2$. According to the formula [5, (4.4)], it follows that the Fourier coefficients of the p -sine function are such that

$$|a_j(p)| \leq \frac{16\pi_p^2 c_p}{\pi^3} j^{-3} \quad \forall j \in \mathbb{N}.$$

Then, $\sin_p(\pi_p \cdot) \in H^\rho$ for all $\rho < \frac{5}{2}$.

Numerical estimates for the Sobolev regularity of $\sin_p(\pi_p \cdot)$ for $2 < p < 100$ were reported in [3, Figure 2]. From that picture, one may conjecture that for $p > 3$, $\sin_p(\pi_p \cdot) \notin H^2$. Moreover, the regularity appears to drop asymptotically to $\frac{3}{2}$ for p large. By contrast, it appears that $\sin_p(\pi_p \cdot) \in H^2$ for $2 < p < 3$. The following statement, which is a consequence of Lemma 7, settles this conjecture.

Corollary 3. *For $p > 2$ set $r(p) = p' + \frac{1}{2}$. Then $\sin_p(\pi_p \cdot) \in H^\rho$ for all $0 \leq \rho < r(p)$.*

Proof. According to Lemma 5,

$$|a_j(p)| = \frac{\pi_p}{j\pi} |b_j(p)|.$$

Then, by virtue of Lemma 7,

$$|a_j(p)| \leq \frac{2\pi_p \pi_{p'}}{\pi^3(p-1)} \left[2 + \frac{\pi^2}{2}(p-2) \right] j^{-(p'+1)} \quad \forall j \geq 3.$$

Let $\langle j \rangle^2 = 1 + j^2$. For $\rho < p' + \frac{1}{2}$,

$$\sum_{j=1}^{\infty} \langle j \rangle^{2\rho} |a_j(p)|^2 \leq 2^\rho a_1(p)^2 + c(p) \sum_{\substack{j=3 \\ j \equiv 2^1}}^{\infty} \frac{1}{j^{1+\epsilon(p)}} < \infty$$

where

$$c(p) = \frac{2\pi_p \pi_{p'}}{\pi^3(p-1)} \left[2 + \frac{\pi^2}{2}(p-2) \right] \quad \text{and} \quad \epsilon(p) = 1 - 2\rho + 2p' > 0.$$

Hence $\sin_p(\pi_p \cdot) \in H^\rho$ as claimed. \square

The recent paper [8] includes various intriguing results connected to Corollary 3.

The paper [7]

The recent paper [7] seems to be the only one in the existing literature which conducts an analysis of the basis properties of the p -cosine functions. In the notation of [7] we fix $\alpha = 1$ and $p = q > 1$. The Fourier coefficients of the p -cosine functions are

$$\tau_j(p, p, 1) = b_j(p) \quad \forall j \in \mathbb{N} \cup \{0\}.$$

The condition [7, (2.2)] as well as the criterion for determining whether $\{\cos_p(n\pi_p \cdot)\}_{n=0}^\infty$ is a Schauder basis of L^s are exactly the same as (12). Let us compare some of the results of [7] with those of the present work.

In [7, Proposition 2.5], the estimate [7, (2.20)] is equivalent to the following. There exists $p_0^* = \frac{72(\pi-2)-2\pi^3}{96(\pi-2)-3\pi^3}$, such that

$$(25) \quad \tau_1(p, p, 1) \geq \begin{cases} \frac{\pi(p-1)}{2p-1} - \frac{(\pi-2)(p-1)}{3p-2} & 1 < p < p_0^* \\ \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)} & p_0^* < p < \infty. \end{cases}$$

Here p_0^* satisfies the identity

$$\frac{4p-3}{3p-2} = \frac{\pi^3}{24(\pi-2)}.$$

Note that $p_0^* \approx 1.22$.

Let us consider firstly the regime $1 < p < 2$. From [7, Proposition 2.2] it follows that

$$(26) \quad \sum_{k=1}^{\infty} |\tau_{2k+1}(p, p, 1)| \leq \frac{\pi_p(\pi^2 - 8)}{\pi^2} \quad \forall p \in (1, 2).$$

As $c_p < 1$ whenever $1 < p < 2$ in (6), then (14) is sharper than (26) in this regime.

If $1 < p < p_0^*$, then

$$\frac{\pi_p(\pi^2 - 8)}{\pi^2} > \frac{\pi(p-1)}{2p-1} - \frac{(\pi-2)(p-1)}{3p-2},$$

and no conclusion about the validity of (12) can be derived in this case from (25) and (26). For $p_0^* < p < 2$, on the other hand,

$$\frac{\pi_p(\pi^2 - 8)}{\pi^3} < \frac{p-1}{2p-1} - \frac{\pi^2(p-1)}{24(4p-3)} \iff p \in (p_0^\dagger, 2),$$

where $p_0^\dagger \approx 1.75$. In order to see this, note that π_p is decreasing and $\lim_{p \rightarrow 1^+} \pi_p = \infty$, while the right hand side of this identity is increasing

for $1 < p < 2$. Thus, a combination of [7, Proposition 2.2] and [7, Proposition 2.5], only guarantees that $\{\cos_p(n\pi_p \cdot)\}_{n=0}^\infty$ is a Schauder basis of L^s for $p \in [p_0^\dagger, 2)$ where $p_0^\dagger > \frac{3}{2} > p_0$.

As it turns, it is not possible to deduce from the results of [7] any basis property of the family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^\infty$ in the complementary regime $p > 2$. Here is how the different estimates on the Fourier coefficients compare in this case.

From [7, Proposition 2.4], we gather that

$$(27) \quad \sum_{k=1}^{\infty} |\tau_{2k+1}(p, p, 1)| \leq \frac{2\pi_{p'}}{\pi^2(p-1)} [4 + \pi(p-1)] \left[\left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1 \right].$$

Since

$$4 + \pi(p-1) \geq 2 + \frac{\pi^2}{2}(p-2) \quad \forall p \leq \frac{4 + 2\pi^2 - 2\pi}{\pi^2 - 2\pi},$$

the upper bound (23) is sharper than (27) for $2 \leq p \leq 3$. The latter is the relevant regime in the proof of Theorem 1.

Since $\pi_p < \pi$ for $p > 2$, the lower bound (24) is sharper than [7, (2.19)]. Moreover,

$$\frac{8}{\pi\pi_p} > \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)} \quad \forall p > 2.$$

Hence the estimate (25), which is [7, (2.20)], is also superseded by (24) for $p > 2$.

Acknowledgements

HM was supported by Heriot-Watt University under the Global Platform Scholarships Scheme.

2010 Mathematics Subject Classification

Primary: 42A16. Secondary: 42A65.

Keywords

Generalised cosine functions, Schauder basis properties, p -Laplacian.

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